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The effects of capillarity on free-streamline separation

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The effect of a small surface-tension coefficient on the classical theory of free-streamline separation from a sharp trailing edge is studied. The classical solution fails in a small region surrounding the edge, where it predicts singular behaviour, and an inner solution, satisfying linear boundary conditions, is required to obtain a uniformly valid first approximation. The solution valid near the edge removes the curvature and pressure-gradient singularities of the classical solution and predicts a standing capillary wave along the free streamline.

1. Introduction

The mathematical theory of free-streamline flows which was developed in the second half of the nineteenth century by Helmholtz and Kirchhoff failed to take into account any effects which capillarity might have on the motion or shape of the free streamline. In 1891 Zhukovskii (see Gurevich 1965, p. 549) included surface-tension effects by deriving a nonlinear boundary condition along the free streamline, which he used in solving for the flow past a bubble; this example was also considered by McLeod (1955). More recently Gurevich (1961) studied the effect that small capillary forces would have on the coefficient of contraction of a jet using a regular perturbation technique, with the Helmholtz potential flow as the first-order approximation.

In this paper we consider how a small surface-tension coefficient modifies the flow in a neighbourhood of a sharp trailing edge to which a free streamline is attached. This is an interesting and important problem because the classical solutions have the following deficiencies.

(i) The curvature of the free streamline at the separation point is infinite and, according to Laplace's capillarity equation (2.7) below, this requires an infinitely negative pressure just inside the fluid along the free streamline.

(ii) If the fluid speed at the separation point is non-zero, the intrinsic equations of motion (see Milne-Thomson 1957, p. 105) predict that the component of the pressure gradient normal to the free streamline is infinite.

(iii) The favourable pressure gradient along the wall upstream of the separation point tends to infinity at the separation point, and the resulting boundary-layer motion, studied by Ackerberg (1970, 1973*a*), implies an infinite skin friction at the edge.

Since there are no reports of bent or broken edges when these flows occur, these predictions must be questioned and interpreted properly.

In many cases of practical interest the fluids under consideration are characterized by small surface-tension coefficients, which, in non-dimensional form, correspond to large Weber numbers. We should anticipate that the difficulties mentioned above are indicative of a local failure of the classical theory (for the limiting case of zero surface tension) in a small neighbourhood of the edge where the curvature is large. Thus, a non-uniform limit is expected near the edge and the inclusion of a small non-zero surface tension as a singular perturbation might remove the singular behaviour. Away from this (inner) region, the classical results should apply to first order. This paper will be concerned with finding first-order solutions using the method of matched asymptotic expansions; there should be no difficulty in principle in extending this analysis to higher orders.

In § 2 the problem is formulated mathematically in the plane of the complex velocity potential using the logarithm of the speed and the deflexion angle as dependent variables. A nonlinear boundary condition is derived along the free streamline which relates the logarithm of the speed to its normal derivative. Denoting the reciprocal of the Weber number by ϵ, it is shown that the limit $\epsilon \to 0$ is non-uniform near the edge owing to the curvature singularity, and an inner region, centred on the edge and of physical dimensions $x = O(\epsilon L)$ by $y = O(\epsilon^{\frac{3}{2}} L)$, where L is a length characterizing the potential flow, is required to obtain a uniformly valid solution. The flow in the inner region satisfies Laplace's equation with linear boundary conditions.

The non-uniqueness of the solution is discussed in § 3, and a fundamental solution is found using the Wiener–Hopf technique. Since we did not know what type of singularity to expect at the separation point we sought the smoothest possible solution. It is remarkable that only one solution exists for which the speed and its normal derivative are continuous at the separation point, and in § 4 this solution is determined in terms of the fundamental solution. Our results indicate a standing capillary wave along the free streamline, and the singularities predicted at the edge by the classical solution are removed. In § 5 numerical solutions for the free-streamline shapes in the inner region are found, and the results are summarized and discussed.

2. Mathematical formulation

Introduce a co-ordinate system $\bar{z} = \bar{x} + i\bar{y}$ with origin at the separation point S (see figure 1). We have chosen the tangent to the wall at S to be parallel to the $\bar{x}$ axis without loss of generality. The velocity components in the directions of $\bar{x}$ and $\bar{y}$ increasing are denoted by $\bar{u}$ and $\bar{v}$, respectively, and a constant gravitational force g (per unit mass) acts in the direction of $\bar{y}$ *decreasing*. The free streamline SB is bounded on one side by a region of constant pressure $\bar{p} = p_0$ in which the fluid is assumed to be at rest, and only motions which are steady and irrotational are considered. Denoting dimensional variables by bars, we introduce the following non-dimensional variables:

$$\left.\begin{aligned} x = \bar{x}/L, \quad y = \bar{y}/L, \quad u = \bar{u}/U_0, \quad v = \bar{v}/U_0, \quad p = (\bar{p} - p_0)/(\tfrac{1}{2}\rho U_0^2), \\ \psi = \bar{\psi}/LU_0, \quad \phi = \bar{\phi}/LU_0, \quad q = \bar{q}/U_0. \end{aligned}\right\} \qquad (2.1)$$

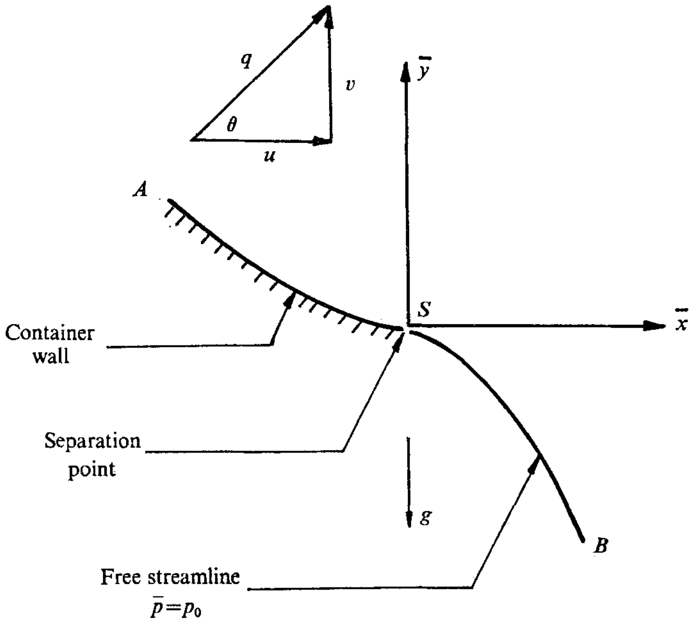

FIGURE 1. Flow geometry.

Here L is a length scale characteristic of the potential flow (e.g. a slot width or plate breadth), U_0 is the fluid speed which would exist at the separation point S *with zero surface-tension coefficient*, ρ is the constant fluid density, $\bar{q} = (\bar{u}^2+\bar{v}^2)^{\frac{1}{2}}$ is the fluid speed and ϕ and ψ are the velocity potential and stream function, which are related to the velocity components by the equations

$$u = \phi_x = \psi_y, \quad v = \phi_y = -\psi_x, \tag{2.2}$$

with subscripts denoting partial differentiation.

Free-streamline problems are characterized by a free boundary whose position in the physical plane is not known in advance. To deal with this problem it is convenient to introduce ϕ and ψ as independent variables and to define the complex velocity potential $w = \phi + i\psi$ and the complex velocity

$$dw/dz = u - iv = q\,e^{-i\theta},$$

where θ is the fluid deflexion angle shown in figure 1. Each of these functions will be analytic in z and the logarithm of the complex velocity, defined by

$$\Gamma(w) \equiv Q(\phi, \psi) - i\theta(\phi, \psi) = \ln(dw/dz), \tag{2.3}$$

with $Q = \ln q$, will be an analytic function of w with Q and θ related by the Cauchy–Rieman equations

$$Q_\psi = \theta_\phi, \quad Q_\phi = -\theta_\psi. \tag{2.4}$$

Thus each function is harmonic in the ϕ, ψ plane and satisfies

$$\nabla^2 Q = \nabla^2\theta = 0, \quad \text{where} \quad \nabla^2 = \partial^2/\partial\phi^2 + \partial^2/\partial\psi^2. \tag{2.5}$$

The flow region in the z plane will be mapped into the upper half of the w plane $(-\infty < \phi < \infty, \psi \geqslant 0)$; the correspondence is shown in figure 2.

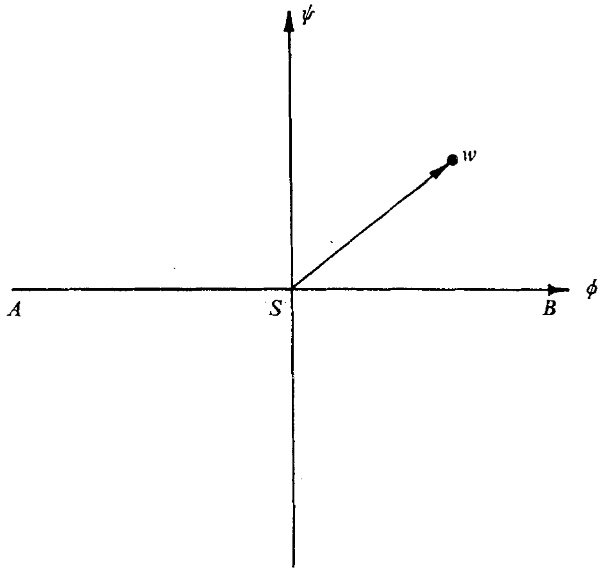

FIGURE 2. The $w = \phi + i\psi$ plane.

Boundary condition along the free streamline

We apply Bernoulli's equation along the free streamline SB in the form

$$\bar{p} + \tfrac{1}{2}\rho\bar{q}^2 + \rho g\bar{y} = p_0 + \tfrac{1}{2}\rho U_0^2 \quad \text{on} \quad \psi = 0, \quad \phi > 0. \tag{2.6}$$

We are free to specify the total head in the plane $\bar{y} = 0$ and have chosen it equal to that which would exist without surface-tension effects. The definitions of p_0 and U_0 given previously are consistent with this choice. It is important to realize that this specification does not require the speed at the separation point to be U_0. The separation speed will depend on the free-streamline curvature at the separation point, as we shall see.

A relationship between $\bar{p}$ and p_0 is given by Laplace's formula

$$\bar{p} - p_0 = \gamma/\bar{R}_f, \tag{2.7}$$

where the surface-tension coefficient γ is expressed in dynes/cm or equivalent units and $R_f = \bar{R}_f/L$ is the non-dimensional radius of curvature of the free streamline. Noting that $(u, v) = \nabla\phi$ and $q = \partial\phi/\partial s$, we have $R_f^{-1} = \partial\theta/\partial s = q\,\partial\theta/\partial\phi$, where s is the non-dimensional arc length measured from the separation point downstream along the free streamline; from these definitions, the curvature of the streamline shown in figure 1 is negative.

Combining (2.6) and (2.7), and introducing the non-dimensionalization (2.1), we obtain

$$\partial\theta/\partial\phi = \tfrac{1}{2}\alpha(q^{-1} - q) - \beta y/q \quad \text{on} \quad \psi = 0, \quad \phi > 0, \tag{2.8}$$

where $\alpha = \rho U_0^2 L/\gamma$ is the Weber number and $\beta = \rho g L^2/\gamma$ is the Bond number. Putting $q = e^Q$ and using (2.4), we may write (2.8) in the form

$$\partial Q/\partial\psi = -\alpha \sinh Q - \beta e^{-Q} y \quad \text{on} \quad \psi = 0, \quad \phi > 0. \tag{2.9}$$

At the separation point, where $y = 0$, (2.8) defines the separation speed q_s in terms of the radius of curvature R_{fs}, i.e.

$$q_s = (1 - 2/\alpha R_{fs})^{\frac{1}{2}}. \tag{2.10}$$

The boundary condition (2.8) was derived by Zhukovskii (see Gurevich 1965, p. 550), and the form (2.9), which relates Q and its normal derivative along the boundary, was obtained by Crapper (1957). It is remarkable that, when $\beta = 0$ (zero gravity), the condition (2.9) is the same as that obtained by Ackerberg & Pal (1968) and Ackerberg (1968*a*, *b*) for the motion along a vortex sheet separating an injected jet from a uniform stream.† Thus a similarity exists between a vortex sheet and a surface exhibiting capillary effects.

Boundary condition on the wall

Let $s_<$ denote arc length measured upstream along the wall from the separation point and suppose that along the wall we may write down the Taylor series expansion

$$\theta(s_<) = \frac{\partial\theta}{\partial s_<}\bigg|_{s_<=0+} s_< + \frac{1}{2}\frac{\partial^2\theta}{\partial s_<^2}\bigg|_{s_<=0+} s_<^2 + \ldots \quad \text{on} \quad \psi = 0, \quad \phi < 0, \tag{2.11}$$

where, by our choice of co-ordinate systems, $\theta(s_< = 0+) = 0$. To relate $s_<$ to ϕ (< 0) along $\psi = 0$, we note that $q_< = \exp Q_< = -\partial\phi/\partial s_<$; thus

$$s_<(\phi) = -\int_0^{\phi} \exp\{-Q(t,0)\}\,dt \quad \text{on} \quad \psi = 0, \quad \phi < 0. \tag{2.12}$$

Henceforth we denote the limiting wall curvature by $\kappa = [\partial\theta/\partial s_<]_{s_<=0+}$ and its derivatives with respect to $s_<$ for $s_< = 0+$ by κ', κ'', etc. In solving a problem with a given geometry (2.11) is not very useful because the functional relationship $s_<(\phi)$ depends on the solution $Q_<$ via (2.12). In the case of a straight wall, (2.11) reduces to

$$\theta(s_<) = 0 \quad \text{on} \quad \psi = 0, \quad \phi < 0. \tag{2.13}$$

We shall be focusing our attention on cases where $\alpha \to \infty$ (i.e. small surface tensions) and (2.11) will suffice for these problems.

Finally, an additional boundary condition at large distances is required to fix the solution of Laplace's equation; this will be derived later.

Parameter sizes and simplification of free-streamline boundary condition

We now assume that the Weber number α is very large and introduce the small parameter

$$\epsilon = \alpha^{-1} \quad (0 < \epsilon \ll 1). \tag{2.14}$$

Later it will be convenient to choose the Bond number according to the relationship

$$0 < \beta\epsilon^2 = \gamma g/\rho U_0^4 \ll 1; \tag{2.15}$$

this is not very restrictive in most physical applications.

† It should be noted, however, that the normal derivatives are of opposite sign since the problem here is formulated in the upper half-plane while the vortex-sheet problem is in a lower half-plane.

To motivate our discussion and simplify the arguments we assume zero gravity, so that $\beta = 0$.† If we formally multiply (2.9) by ϵ and then put $\epsilon \equiv 0$, we obtain

$$\sinh Q = 0 \quad \text{on} \quad \psi = 0, \quad \phi > 0, \tag{2.16}$$

with solution

$$Q = 0 \quad \text{on} \quad \psi = 0, \quad \phi > 0. \tag{2.17}$$

This is the classical free-streamline boundary condition and the first-order outer problem ($\epsilon \equiv 0$) is simply that of potential flow with surface tension neglected. We assume that the behaviour of this outer solution near the separation point S is given by

$$\Gamma(w) = Q - i\theta = A_0(e^{-\pi i}w)^{\frac{1}{2}} \quad \text{for} \quad |w| \to 0 \quad (\pi \geqslant \arg w \geqslant 0), \tag{2.18}$$

where the constant $A_0 < 0$. The result (2.18) and the sign of A_0 correspond to the abrupt separation considered by Carter (1961) and Ackerberg (1970, 1973 *a*, *b*). For these cases the limit $\epsilon \to 0$, which yields (2.16), cannot be uniformly valid for (2.18) predicts that

$$\partial Q/\partial \psi = \partial \theta/\partial \phi = \tfrac{1}{2}A_0\phi^{-\frac{1}{2}} \quad \text{for} \quad \phi \to 0+ \quad \text{on} \quad \psi = 0, \tag{2.19}$$

and $\epsilon\, \partial Q/\partial \psi$ can be made to dominate the term $\sinh Q$ if ϕ is chosen sufficiently small [see (2.9)]. The remedy is to introduce an inner region near the separation point in which the terms in (2.9) balance. One might expect $\phi = O(\epsilon^2)$ in this region, but this is erroneous because $Q = o(1)$ near the separation point.

Scaling for the inner region

Our interest is in the region where $w = o(1)$ and we expect $\Gamma = Q - i\theta = o(1)$; thus we introduce new variables

$$w = \epsilon^a w^*, \quad \Gamma(w) = \epsilon^b \Gamma^*(w^*), \tag{2.20}$$

and anticipate that $a, b > 0$ and w^* and Γ^* remain $O(1)$. Since the Cauchy–Riemann equations are linear in w, Q and θ, they remain unchanged by these transformations and (2.5) requires

$$\nabla_*^2 \theta^* = \nabla_*^2 Q^* = 0, \quad \text{where} \quad \nabla_*^2 = \partial^2/\partial\phi^{*2} + \partial^2/\partial\psi^{*2}. \tag{2.21}$$

To satisfy the boundary conditions, we substitute (2.20) into (2.9) and expand $\sinh Q$ and e^{-Q} for $\epsilon \to 0$; thus

$$\partial Q^*/\partial \psi^* = -\epsilon^{a-1}Q^* - \beta\epsilon^{a-b}y \quad \text{on} \quad \psi^* = 0, \quad \phi^* > 0. \tag{2.22}$$

To estimate the size of y in this region, we use (2.3) and (2.20) to write

$$z = \int_0^w \exp\{-\Gamma(s)\}\, ds = \epsilon^a \int_0^{w^*} \exp\{-\epsilon^b \Gamma^*(t^*)\}\, dt^*$$

$$= \epsilon^a \int_0^{w^*} \exp\{-\epsilon^b Q^*\}[\cos(\epsilon^b\theta^*) + i\sin(\epsilon^b\theta^*)]\, dt^*. \tag{2.23}$$

Expanding for $\epsilon \to 0$ we find

$$x = O(\epsilon^a) \quad \text{and} \quad y = O(\epsilon^{a+b}), \tag{2.24}$$

† This assumption will be removed later.

and for later use

$$s = \int_0^x [1+(dy/dx)^2]^{\frac{1}{2}}\,dx = O(\epsilon^a). \tag{2.25}$$

Using (2.24), the first term in (2.22) dominates the third provided that $\beta\epsilon^{2a} = o(1)$, and to achieve a balance between the first and second terms we choose $a = 1$ to obtain

$$\partial Q^*/\partial\psi^* = -Q^* \quad \text{on} \quad \psi^* = 0, \quad \phi^* > 0. \tag{2.26}$$

The physical significance of the elimination of the third term in (2.22) is that the gravitational force does not produce any first-order effects in the region given by (2.24) provided that (2.15) is satisfied.

The most useful boundary condition along the wall AS is obtained by considering the differentiated form of (2.11), i.e.

$$\partial\theta/\partial s_< = -q_<\partial\theta/\partial\phi = \kappa+\kappa' s_< + \dots \quad \text{on} \quad \psi^* = 0, \quad \phi^* < 0, \tag{2.27}$$

where we have used (2.12). Introducing the change of variables (2.20) we find

$$\partial\theta^*/\partial\phi^* = -\epsilon^{a-b}\exp(-\epsilon^b Q^*)(\kappa+\kappa' s_< + \dots) \quad \text{on} \quad \psi^* = 0, \quad \phi^* < 0. \tag{2.28}$$

Expanding (2.28) for $\epsilon \to 0$ and using (2.25) we obtain

$$\partial\theta^*/\partial\phi^* = -\epsilon^{a-b}\kappa + O(\epsilon^a, \epsilon^{2a-b}) \quad \text{on} \quad \psi^* = 0, \quad \phi^* < 0, \tag{2.29}$$

assuming that the non-dimensional curvature and its derivatives are all $O(1)$.† Putting $a = 1$ and anticipating the result in the next paragraph, where we find $b = \frac{1}{2}$, we obtain in the limit $\epsilon \to 0$

$$\partial\theta^*/\partial\phi^* = \partial Q^*/\partial\psi^* = 0 \quad \text{on} \quad \psi^* = 0, \quad \phi^* < 0, \tag{2.30}$$

using the Cauchy–Riemann equations. The significance of this result is that, within the region $w^* = O(1)$, the wall appears flat, to first order, with infinite radius of curvature.

To complete the formulation of the boundary-value problem for the inner region, we require a condition for $|w^*| \to \infty$. The region $|w^*| = O(1)$ is embedded within a potential flow where (2.18) is valid for $|w| \to 0$. Therefore, we require $Q^* - i\theta^*$ to asymptote to (2.18) for $|w^*| \to \infty$. Expressing (2.18) in terms of the variables (2.20) we find

$$Q^* - i\theta^* \sim \epsilon^{b-\frac{1}{2}a} A_0 (e^{-\pi i} w^*)^{\frac{1}{2}} \quad \text{for} \quad |w^*| \to \infty \quad \text{with} \quad \psi^* > 0. \tag{2.31}$$

Noting that (2.21) and the boundary conditions (2.26) and (2.30) admit a trivial solution, we introduce a forcing term via (2.31) by choosing

$$b = \tfrac{1}{2}a = \tfrac{1}{2}. \tag{2.32}$$

Collecting our results and suppressing the asterisks, we have the following boundary-value problem:

$$\partial^2 Q/\partial\phi^2 + \partial^2 Q/\partial\psi^2 = 0 \quad \text{for} \quad -\infty < \phi < \infty, \quad \psi > 0, \tag{2.33}$$

$$\partial Q/\partial\psi = \begin{cases} 0 & \text{on} \quad -\infty < \phi < 0, \\ -Q & \text{on} \quad 0 < \phi < \infty, \end{cases} \quad \psi = 0, \tag{2.34), (2.35}$$

$$Q \sim \operatorname{Re}\{A_0 (e^{-\pi i} w)^{\frac{1}{2}}\} \quad \text{for} \quad |w| \to \infty, \quad \arg w > 0. \tag{2.36}$$

† This restriction is stronger than necessary and could be relaxed.

The condition $\arg w > 0$ in (2.36) will not be superfluous because we shall find a standing capillary wave along the free streamline $\psi \equiv 0$ which is not attenuated as $\phi \to \infty$. It should also be noted that an edge condition at $w = 0$ is frequently specified for split boundary-value problems of this type. Since we are not certain what continuity conditions to expect, we simply require the solution and its normal derivative to be as smooth as possible at the edge; this will be discussed later.

3. An integral equation and its solution by the Wiener–Hopf technique

A standard method for solving a potential problem such as (2.33)–(2.36) is to use a Green's function to formulate it as an integral equation, which is solved using the Wiener–Hopf technique. A straightforward application of this method fails, however, because the solution sought is unbounded for $|w| \to \infty$. We may still employ the standard method if we temporarily overlook (2.36) and note that any ϕ derivative of Q will also satisfy (2.33)–(2.35). Therefore, we shall solve (2.33)–(2.35), as described, for a function $U(\phi, \psi)$ which we shall designate as the fundamental solution, and then determine a solution for $Q(\phi, \psi)$ using ϕ integrals and ϕ derivatives of $U(\phi, \psi)$.

To cast the problem into a convenient form, we consider the slightly more general equation

$$\nabla^2 U - k^2 U = 0, \tag{3.1}$$

where k is a positive parameter which will be allowed to approach zero at a convenient point in the analysis and $U(\phi, \psi)$ will be subjected to the boundary conditions (2.34) and (2.35). The appropriate Green's function which has a zero normal derivative along $\psi = 0$ is

$$G(\phi, \psi | s, t) = -(2\pi)^{-1}\{K_0(k[(\phi - s)^2 + (\psi - t)^2]^{\frac{1}{2}}) + K_0(k[(\phi - s)^2 + (\psi + t)^2]^{\frac{1}{2}})\}, \tag{3.2}$$

where K_0 is the modified Bessel function of the second kind of order zero. If we apply Green's theorem and allow for the boundary conditions (2.34) and (2.35) we obtain

$$U(\phi, 0) = \pi^{-1} \int_0^\infty U(s, 0)\, K_0(k|\phi - s|)\, ds \quad \text{for} \quad -\infty < \phi < \infty. \tag{3.3}$$

In obtaining (3.3) it has been assumed that no contributions arise from contour integrals for $|w| \to \infty$ and $|w| \to 0$; this can be verified *a posteriori*.

The Wiener–Hopf technique

Define the new functions

$$f(\phi) = \begin{cases} U(\phi, 0) & \text{for} \quad \phi > 0, \\ 0 & \text{for} \quad \phi < 0, \end{cases} \tag{3.4}$$

and

$$g(\phi) = \begin{cases} 0 & \text{for} \quad \phi > 0, \\ U(\phi, 0) & \text{for} \quad \phi < 0. \end{cases} \tag{3.5}$$

The integral equation (3.3) may now be written as

$$f(\phi)+g(\phi) = \pi^{-1}\int_{-\infty}^{\infty} f(s)\,K_0(k|\phi-s|)\,ds. \tag{3.6}$$

Introduce the Fourier transform and its inverse

$$F_-(\omega) = \int_{-\infty}^{\infty} e^{-i\phi\omega} f(\phi)\,d\phi, \quad f(\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\phi\omega} F_-(\omega)\,d\omega. \tag{3.7}$$

From the definition (3.4), $F_-(\omega)$ will be an analytic function in the lower half-plane; this is denoted by the suffix minus. Similarly $G_+(\omega)$, the Fourier transform of $g(\phi)$, will be an analytic function in the upper half-plane. On taking the Fourier transform of (3.6), we obtain

$$F_-(\omega)+G_+(\omega) = \pi^{-1}\bar{K}(\omega)\,F_-(\omega), \tag{3.8}$$

where

$$\bar{K}(\omega) = \pi(k^2+\omega^2)^{-\frac{1}{2}} \tag{3.9}$$

is the Fourier transform of $K_0(k\,|\phi|)$ and is a regular function in the ω plane cut along the imaginary axis from ik to $i\infty$ and $-i\infty$ to $-ik$. Rewriting (3.8) we find

$$F_-(\omega)\,[1-(k^2+\omega^2)^{-\frac{1}{2}}] = -G_+(\omega). \tag{3.10}$$

In applying the Wiener–Hopf technique it is necessary to write the coefficient of $F_-(\omega)$ as the ratio of two functions, the numerator being analytic in the lower half-plane and the denominator being analytic in the upper half-plane. To carry out this splitting we note that the bracketed term in (3.10) has zeros when

$$\omega = \pm\lambda, \quad \text{where} \quad \lambda = (1-k^2)^{\frac{1}{2}}. \tag{3.11}$$

We explicitly display these zeros by writing

$$1-(k^2+\omega^2)^{-\frac{1}{2}} = (\omega^2-\lambda^2)\,[(k^2+\omega^2)^{\frac{1}{2}}-1]/(\omega^2-\lambda^2)\,(k^2+\omega^2)^{\frac{1}{2}} \tag{3.12}$$

and carry out the split

$$L(\omega) \equiv [1-(k^2+\omega^2)^{\frac{1}{2}}]/(\omega^2-\lambda^2)\,(k^2+\omega^2)^{\frac{1}{2}} = L_-(\omega)/L_+(\omega), \tag{3.13}$$

where according to Noble (1958, pp. 15 ff.)

$$\ln L_+(\omega) = (2\pi i)^{-1}\int_{-\infty+ic}^{\infty+ic} (\omega-t)^{-1}\ln[L(t)]\,dt, \tag{3.14}$$

$$\ln L_-(\omega) = (2\pi i)^{-1}\int_{-\infty+id}^{\infty+id} (\omega-t)^{-1}\ln[L(t)]\,dt \tag{3.15}$$

and $-k < c < \operatorname{Im}\omega < d < k$. To evaluate these integrals it is convenient first to differentiate with respect to ω and then displace the contours such that the integral for L_- (say) embraces the branch cut from $i\infty$ to ik. Taking into account the contribution from the neighbourhood of the branch point ik, we find after letting $k \to 0$

$$L'_-(\omega)/L_-(\omega) = -(2\omega)^{-1}-\frac{i}{\pi}\int_0^{\infty} [(t+i\omega)\,(1+t^2)]^{-1}\,dt.\dagger \tag{3.16}$$

† Primes denote differentiation with respect to ω.

To complete the integration, let ω be a point on the negative imaginary axis and put $\omega = -is$ with $s > 0$. We find

$$L_-(-is) = C[s^2(1+s^2)]^{-\frac{1}{4}} \exp\left\{\pi^{-1}\int_0^s (1+t^2)^{-1}\ln t\,dt\right\}, \tag{3.17}$$

where C is a constant of integration. To determine $L_-(\omega)$ elsewhere, introduce a cut along the positive imaginary axis. Thus, if ω is real and positive,

$$L_-(\omega) = C\exp\left(-\tfrac{1}{4}\pi i\right)[\omega(1+\omega)]^{-\frac{1}{2}} \exp\left\{(i/\pi)\int_0^\omega (1-s^2)^{-1}\ln s\,ds\right\}. \tag{3.18}$$

Similarly, if $\omega = is$ with s real and positive

$$L_+(is) = -C[s^2(1+s^2)]^{\frac{1}{4}} \exp\left\{-\pi^{-1}\int_0^s (1+t^2)^{-1}\ln t\,dt\right\}, \tag{3.19}$$

and if ω is real and positive

$$L_+(\omega) = -C\exp\left(-\tfrac{1}{4}\pi i\right)[\omega(1+\omega)]^{\frac{1}{2}} \exp\left\{(i/\pi)\int_0^\omega (1-s^2)^{-1}\ln s\,ds\right\}. \tag{3.20}$$

In obtaining (3.19) and (3.20) a constant of integration was chosen such that (3.13) is satisfied when $k = 0$ and ω is real.

If we assume, for the moment, that $k \neq 0$ and substitute (3.13) into (3.10) we obtain

$$F_-(\omega)\,L_-(\omega)\,(\omega^2-\lambda^2) = G_+(\omega)\,L_+(\omega). \tag{3.21}$$

We now assert that F_- and G_+ are analytic in the strip $0 > \operatorname{Im}\omega > -k$ and that the left-hand side of (3.21) defines a function which is analytic in the lower half-plane $\operatorname{Im}\omega < 0$. Similarly the right-hand side is analytic in the upper half-plane $\operatorname{Im}\omega > -k$. It follows that both sides must equal a function which is analytic everywhere, except possibly at the point at infinity. We shall assume (and verify *a posteriori*) that the correct analytic function is a constant A; therefore

$$F_-(\omega) = A/[(\omega^2-\lambda^2)\,L_-(\omega)], \quad G_+(\omega) = A/L_+(\omega). \tag{3.22}$$

To complete the determination of f and g we use the inversion integral (3.7) to write

$$f(\phi) = (2\pi)^{-1}A\int_{-\infty}^{\infty} [(\omega^2-1)\,L_-(\omega)]^{-1}\,e^{i\phi\omega}\,d\omega \tag{3.23}$$

and

$$g(\phi) = (2\pi)^{-1}A\int_{-\infty}^{\infty} [L_+(\omega)]^{-1}\,e^{i\phi\omega}\,d\omega, \tag{3.24}$$

where we have now put $k = 0$ so that $\lambda^2 = 1$. Both integrals are evaluated using contour integration, which for the case of $g(\phi)$ involves deforming the contour such that it embraces both sides of the negative imaginary axis and then using (3.13) and (3.17) to find the values of $L_+(\omega)$ on either side of the cut. The details are messy and we simply give the results:

$$g(\phi) = -(A/C)\,\pi^{-1}\int_0^\infty \exp\{\phi t - H(t)\}\,t^{-\frac{1}{2}}(1+t^2)^{-\frac{3}{4}}\,dt \quad \text{for} \quad \phi \leqslant 0 \tag{3.25}$$

and

$$f(\phi) = (A/C)\left\{-2^{\frac{1}{4}}\sin\left(\phi+\tfrac{3}{8}\pi\right)+\pi^{-1}\int_0^\infty \exp\{-\phi t+H(t)\}\,t^{\frac{1}{2}}(1+t^2)^{-\frac{5}{4}}\,dt\right\} \quad \text{for} \quad \phi \geqslant 0, \tag{3.26}$$

where

$$H(t) = \pi^{-1}\int_0^t (1+s^2)^{-1}\ln s\,ds \tag{3.27}$$

and the sine term in (3.26) is a result of the poles in (3.23) at $\omega = \pm 1$.

Once $f(\phi)$ is known, it is a simple matter to determine $U(\phi, \psi)$ throughout the upper half-plane. This is done by taking the Fourier transform of (3.1) with respect to ϕ, solving the resulting differential equation to obtain a solution which is bounded for $\psi \to \infty$, and determining the remaining constant in the solution using the Fourier transforms of the boundary conditions (2.34) and (2.35) and the transform of $f(\phi)$ given by (3.22). After using the inversion integral we obtain

$$U(\phi, \psi) = (A/2\pi)\int_{-\infty}^{\infty} \frac{\exp\{i\phi\omega - (\omega^2+k^2)^{\frac{1}{2}}\psi\}}{(\omega^2-\lambda^2)(\omega^2+k^2)^{\frac{1}{2}}L_-(\omega)}\,d\omega, \tag{3.28}$$

or

$$U(\phi, \psi) = (A/2\pi)\int_{-\infty}^{\infty} \frac{\exp\{i\phi\omega - (\omega^2+k^2)^{\frac{1}{2}}\psi\}}{[1-(k^2+\omega^2)^{\frac{1}{2}}]L_+(\omega)}\,d\omega. \tag{3.29}$$

These integrals are evaluated using contour integration. For $\phi < 0$ we use (3.28) and (3.17) and close the contour in the lower half-plane, while for $\phi > 0$ we use (3.29) and (3.19) and close the contour in the upper half-plane. The following results are obtained after letting $k \to 0$:

$$U(\phi, \psi) = -(A/C)\,\pi^{-1}\int_0^{\infty} \exp\{\phi t - H(t)\}\,t^{-\frac{1}{2}}(1+t^2)^{-\frac{3}{4}}\cos(\psi t)\,dt \quad \text{for} \quad \phi \leqslant 0, \quad \psi \geqslant 0 \tag{3.30}$$

and

$$U(\phi, \psi) = (A/C)\Big\{-2^{\frac{1}{2}}e^{-\psi}\sin(\phi + \tfrac{3}{8}\pi) + \pi^{-1}\int_0^{\infty}\exp\{-\phi t + H(t)\}\,t^{-\frac{1}{2}}(1+t^2)^{-\frac{5}{4}}(t\cos\psi t - \sin\psi t)\,dt\Big\} \quad \text{for} \quad \phi \geqslant 0, \quad \psi \geqslant 0. \tag{3.31}$$

It can be verified directly that (3.30) and (3.31) are solutions of (2.33)–(2.35). The most interesting aspect of the solution is the appearance of a standing capillary wave on $\psi = 0$, $\phi > 0$ (i.e. the free streamline) which decays exponentially for $\psi \to \infty$. We now observe that any ϕ derivative of (3.30) and (3.31) generates a new solution of (2.33)–(2.35), and thus our boundary-value problem *does not possess a unique solution* without some additional constraints.

Asymptotic expansions for $|\phi| \to 0$ *and* $|\phi| \to \infty$

Asymptotic expansions for $|\phi| \to 0$ can be found by expanding $F_-(\omega)$ and $G_+(\omega)$ for $|\omega| \to \infty$ and interpreting each term separately. A very convenient table to use for this purpose has been compiled by Geller (1963). If we substitute (3.18) and (3.20) into (3.22) and expand for $\omega \to \infty$ we find

$$G_+(\omega) \sim -(Ai/C\omega)\{1 - (i/\pi\omega)(\ln\omega + 1) - (2\omega)^{-1} + O[(\ln\omega/\omega)^2]\}, \tag{3.32}$$

and

$$F_-(\omega) \sim (Ai/C\omega)\{1 - (i/\pi\omega)(\ln\omega + 1) + (2\omega)^{-1} + O[(\ln\omega/\omega)^2]\}. \tag{3.33}$$

Thus

$$g(\phi) \sim -(A/C)\{1 + \pi^{-1}|\phi|(\ln|\phi| + \gamma - 2) + O[(\phi\ln|\phi|)^2]\} \quad \text{for} \quad \phi \to 0-, \tag{3.34}$$

$$f(\phi) \sim -(A/C)\{1 - \pi^{-1}\phi(\ln\phi + \gamma - 2) + O[(\phi\ln\phi)^2]\} \quad \text{for} \quad \phi \to 0+, \tag{3.35}$$

where Euler's constant $\gamma = 0{\cdot}5772\ldots$. We note that $U(\phi, 0)$ is continuous for $|\phi| \to 0$, but it may be seen from (2.34) and (2.35) that its normal derivative $[\partial U/\partial\psi]_{\psi=0}$ is not, since $U(\phi, 0) = f(\phi) \neq 0$ for $\phi \to 0+$.

Asymptotic expansions can be found for $|\phi| \to \infty$ by expanding the integrands of (3.25) and (3.26) for $t \to 0$ and integrating term by term. In this way we obtain (see Geller 1963)

$$f(\phi) \sim -(2\pi^{-\frac{1}{2}})(A/C)\{(\tfrac{1}{2}\pi)^{\frac{1}{2}}\sin(\phi+\tfrac{3}{8}\pi)-(4\phi^{\frac{3}{2}})^{-1} + (8\pi\phi^{\frac{5}{2}})^{-1}[3\ln\phi+3(\gamma+2\ln 2)-5]+o(\phi^{-\frac{5}{2}})\} \quad \text{for} \quad \phi\to\infty \tag{3.36}$$

and

$$g(\phi) \sim -(2\pi^{-\frac{1}{2}})(A/C)\{(2|\phi|^{\frac{1}{2}})^{-1}+(4\pi|\phi|^{\frac{3}{2}})^{-1} \times[\ln|\phi|+\gamma+2\ln 2-1]+o(|\phi|^{-\frac{3}{2}})\} \quad \text{for} \quad \phi\to-\infty. \tag{3.37}$$

We have observed that the sine term in (3.31) and (3.36) is only important on the boundary $\psi = 0$, $\phi > 0$, where it is not exponentially small. If we require $\arg w > 0$, it is easy to infer from (3.36) and (3.37) that the asymptotic form of $U(\phi, \psi)$ will be

$$U(\phi,\psi) \sim -(2\pi^{-\frac{1}{2}})(A/C)\,\mathrm{Re}\{\tfrac{1}{2}(e^{-\pi i}w)^{-\frac{1}{2}}+(4\pi)^{-1}(e^{-\pi i}w)^{-\frac{3}{2}} \times[\ln(e^{-\pi i}w)+\gamma+2\ln 2-1]+o(w^{-\frac{3}{2}})\} \quad \text{for} \quad |w|\to\infty \quad (\pi \geqslant \arg w > 0). \tag{3.38}$$

On comparing (3.38) with (2.36) we might expect to find that $Q(\phi, \psi)$ is the ϕ integral of the fundamental solution $U(\phi, \psi)$.

4. The solution for $Q(\phi, \psi)$

From our previous discussion we expect to find

$$Q(\phi,\psi) = \int_0^{\phi} U(s,\psi)\,ds + M(\psi), \tag{4.1}$$

where $M(\psi)$ is an unknown function to be determined. We now assume (4.1) to be valid and show that the original boundary-value problem (2.33)–(2.36) can be satisfied by this choice. Moreover we shall find that $Q(\phi, \psi)$ and its normal derivative are continuous at $w = 0$, and that it is the only solution possessing these continuity properties. In the discussion which follows, we shall assume that $U(\phi, \psi)$ obeys (3.1) with $k = 0$.

First we verify that (4.1) satisfies Laplace's equation for $\psi > 0$. Differentiating (4.1) twice with respect to ψ we obtain†

$$\frac{\partial^2 Q}{\partial\psi^2} = \int_0^{\phi}\frac{\partial^2 U(s,\psi)}{\partial\psi^2}\,ds + \frac{d^2M}{d\psi^2}. \tag{4.2}$$

Substituting $\partial^2 U/\partial\psi^2 = -\partial^2 U/\partial\phi^2$ and integrating, we find

$$\frac{\partial^2 Q}{\partial\psi^2} = -\frac{\partial U}{\partial\phi} + \left.\frac{\partial U}{\partial\phi}\right|_{\phi=0} + \frac{d^2M}{d\psi^2}. \tag{4.3}$$

From (4.1), $\partial U/\partial\phi = \partial^2 Q/\partial\phi^2$ and (4.3) may be written as

$$\nabla^2 Q = [\partial U/\partial\phi]_{\phi=0} + d^2M/d\psi^2. \tag{4.4}$$

† Differentiating under the integral sign is permissible since U and its derivatives are uniformly continuous for $\psi > 0$. This is a property of harmonic functions.

Therefore Q will satisfy Laplace's equation provided that $M(\psi)$ obeys the second-order differential equation

$$d^2M/d\psi^2 = -[\partial U/\partial\phi]_{\phi=0} \quad \text{for} \quad \psi > 0. \tag{4.5}$$

We now check the boundary conditions (2.34) and (2.35) on $\psi = 0$. Differentiating (4.1) with respect to ψ and taking the limit $\psi \to 0$ yields

$$\left.\frac{\partial Q}{\partial \psi}\right|_{\psi=0} = \int_0^\phi \frac{\partial U}{\partial \psi}(s,0)\,ds + \left.\frac{dM}{d\psi}\right|_{\psi=0+}. \tag{4.6}$$

For $\phi < 0$, $[\partial U/\partial\psi]_{\psi=0} = 0$ and thus (2.34) will be satisfied if

$$dM/d\psi = 0 \quad \text{for} \quad \psi \to 0+.\dagger \tag{4.7}$$

For $\phi > 0$, $[\partial U/\partial\psi]_{\psi=0} = -U(\phi,0)$ and (4.6) requires

$$\left.\frac{\partial Q}{\partial \psi}\right|_{\psi=0} = -\int_0^\phi U(s,0)\,ds = -Q(\phi,0) + M(0+), \tag{4.8}$$

where we have used (4.7) and (4.1). Therefore (2.35) will be satisfied if

$$M = 0 \quad \text{for} \quad \psi \to 0+. \tag{4.9}$$

Thus $M(\psi)$ satisfies (4.5) subject to the initial conditions (4.7) and (4.9). To guarantee that such a function exists we check the behaviour of $[\partial U/\partial\phi]_{\phi=0}$ for $\psi \to 0+$ using the asymptotic expansion of $U(\phi,\psi)$ for $w \to 0$. Using (3.34) and (3.35) we infer that for $w \to 0$

$$U(\phi,\psi) \sim -(A/C)\,\mathrm{Re}\{1 + \pi^{-1}(e^{-\pi i}w)[\ln(e^{-\pi i}w) + \gamma - 2] + \ldots\},$$
$$\text{where} \quad 0 \leqslant \arg w \leqslant \pi. \tag{4.10}$$

Hence $$[\partial U/\partial\phi]_{\phi=0} = O(\ln\psi) \quad \text{for} \quad \psi \to 0+, \tag{4.11}$$

and (4.5) is integrable for $\psi \to 0+$.

Finally, we must verify that (2.36) is satisfied for $|w| \to \infty$ with $\arg w > 0$. Using the complex form (3.38), we can show that

$$Q(\phi,\psi) \sim (2\pi^{-\frac{1}{2}})(A/C)\,\mathrm{Re}\{(e^{-\pi i}w)^{\frac{1}{2}} - (2\pi)^{-1}(e^{-\pi i}w)^{-\frac{1}{2}}$$
$$\times [\ln(e^{-\pi i}w) + \gamma + 2\ln 2 + 1] + \ldots\} \quad \text{for} \quad |w| \to \infty \quad (\pi \geqslant \arg w > 0). \tag{4.12}$$

The first-order term in this equation will agree with (2.36) if we choose

$$2\pi^{-\frac{1}{2}}A/C = A_0. \tag{4.13}$$

Since A_0 is known from the global potential solution with surface tension neglected, the solution given by (4.1) with $M(\psi)$ determined from (4.5), (4.7) and (4.9) is completely defined and satisfies the original boundary-value problem (2.33)–(2.36).

Uniqueness of the solution

If we apply (4.1) at the separation point S, where $\phi = \psi = 0$, we find

$$Q(0,0) = 0. \tag{4.14}$$

This surprising result states that the speed at the separation point is the same (to first order) as for the case with surface tension neglected. Using this result,

† Since $M(\psi)$ satisfies a differential equation valid for $\psi > 0$, the boundary condition should be interpreted as a limiting value for $\psi \to 0+$.

we see from (2.34) and (2.35) that the normal derivative $\partial Q/\partial\psi$ is continuous at S. It was mentioned earlier that the fundamental solution $U(\phi,\psi)$ and all its ϕ derivatives satisfy (2.33)–(2.35) and from (3.38) we see they are $o(w^{\frac{1}{2}})$ for $|w|\rightarrow\infty$; thus these solutions may be considered to be eigenfunctions for this problem. However, the addition of any multiple of $U(\phi,\psi)$ to $Q(\phi,\psi)$ will produce a new solution whose normal derivative is not continuous at S while the addition of any ϕ derivative of U will produce a new solution with Q not bounded at S [see (3.34) and (3.35)]. In physical problems where there is a choice of continuity conditions to be imposed, it is reasonable to choose the most continuous solution, especially when that solution is unique as it is in our case. We therefore specify our solution by (4.1) with the constant A/C given by (4.13). Note that the addition to Q of any ϕ integral of Q will also yield a smooth solution at S, but the boundary condition at infinity (2.36) will not be satisfied.

Matching with the outer solution

For $\arg w > 0$ the matching with the first-order outer flow is provided by (2.36). When subsequent terms in the asymptotic expansion (4.12) are expressed in terms of the outer variables (remembering that w in §§ 3 and 4 is really w^*), they represent higher-order terms in the outer solution. Along the free boundary, where $\psi\equiv 0$ and $\phi>0$, the inner solution introduces a term of the form

$$\epsilon^{\frac{1}{2}}\cos(\epsilon^{-1}\phi+\tfrac{3}{8}\pi). \tag{4.15}$$

Since this term is multiplied by an exponentially small factor in the outer region where $\psi = O(1)$, a standing wave can never enter the outer expansion except as a transcendentally small term.

The functions $Q(\phi,0)$ and $\theta(\phi,0)$

We introduce the notation

$$Q(\phi,0)=\begin{cases}Q_>(\phi) & \text{for}\quad \phi\geqslant 0,\\ Q_<(\phi) & \text{for}\quad \phi\leqslant 0,\end{cases} \tag{4.16}$$

and note that $Q(\phi,0)$ is defined by (3.25), (3.26), (4.1) and (4.9). After carrying out the integrations and noting that $A_0<0$, we obtain

$$\begin{aligned}\Omega(\phi) &\equiv Q_>(\phi)/|A_0|+(\tfrac{1}{2}\pi)^{\frac{1}{2}}\cos(\phi+\tfrac{3}{8}\pi)\\ &=(\tfrac{1}{2}\pi)^{\frac{1}{2}}\cos(\tfrac{3}{8}\pi)+(4\pi)^{-\frac{1}{2}}\int_0^\infty(e^{-\phi s}-1)\,e^{H(s)}s^{-\frac{1}{2}}(1+s^2)^{-\frac{5}{4}}\,ds\\ &\qquad\qquad \text{for}\quad \phi\geqslant 0\end{aligned} \tag{4.17}$$

and

$$Q_<(\phi)/|A_0|=(4\pi)^{-\frac{1}{2}}\int_0^\infty(e^{\phi s}-1)\,e^{-H(s)}s^{-\frac{3}{2}}(1+s^2)^{-\frac{3}{4}}\,ds\quad\text{for}\quad\phi\leqslant 0. \tag{4.18}$$

Using the Cauchy–Riemann equations (2.4) and the boundary condition (2.35) we may write

$$\theta_>(\phi)=-\int_0^\phi Q_>(s)\,ds\quad\text{for}\quad\phi\geqslant 0\quad\text{on}\quad\psi=0. \tag{4.19}$$

Substituting (4.17) into (4.19) and integrating, we find

$$\Lambda(\phi) \equiv \theta_>(\phi)/|A_0| - (\tfrac{1}{2}\pi)^{\frac{1}{2}}\sin(\phi+\tfrac{3}{8}\pi)$$
$$= -(\tfrac{1}{2}\pi)^{\frac{1}{2}}\sin(\tfrac{3}{8}\pi) - c_1\phi + (4\pi)^{-\frac{1}{2}}\int_0^\infty (e^{-\phi s}-1)\,e^{H(s)}s^{-\frac{3}{2}}(1+s^2)^{-\frac{1}{4}}\,ds \quad \text{for} \quad \phi \geqslant 0, \tag{4.20}$$

where
$$c_1 = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\cos(\tfrac{3}{8}\pi) - (4\pi)^{-\frac{1}{2}}\int_0^\infty e^{H(s)}s^{-\frac{1}{2}}(1+s^2)^{-\frac{1}{4}}\,ds \equiv 0. \tag{4.21}$$

The result (4.21) was obtained numerically, but it can be proved analytically by an application of
$$\int_{\mathscr{D}} \frac{\partial U}{\partial n}\,ds = 0, \tag{4.22}$$
where U is the fundamental solution satisfying (2.33)–(2.35) and $\mathscr{D}$ is the boundary of the domain, i.e. the upper half-plane. The result (4.22) follows from applying Green's theorem to (2.33). Some care is required in handling the surface-wave term in (3.31), which contributes to the integral around the infinite semi-circle in the upper half-plane.

Asymptotic expansions of $Q_<$, $Q_>$ and $\theta_>$ for $|\phi|\to 0$ can be obtained directly by using (3.34) and (3.35) with (4.1), (4.9) and (4.19). The procedure is straightforward and these results are not given here.

The asymptotic results for $|\phi|\to\infty$ are more challenging to find because the integrals in (4.17), (4.18) and (4.20) are marginally convergent and some care is necessary in evaluating them. After some extensive algebra we obtained the following results:

$$\Omega(\phi)/|A_0| \sim c_1 + (2\phi^{\frac{1}{2}})^{-1} - (4\pi\phi^{\frac{3}{2}})^{-1}[\ln\phi + \gamma - 1 + 2\ln 2] + o(\phi^{-\frac{3}{2}}) \quad \text{for} \quad \phi\to\infty, \tag{4.23}$$
$$Q_<(\phi)/|A_0| \sim -|\phi|^{\frac{1}{2}} + c_2 + (2\pi|\phi|^{\frac{1}{2}})^{-1}[\ln|\phi| + \gamma + 1 + 2\ln 2] + o(|\phi|^{-\frac{1}{2}}) \quad \text{for} \quad \phi\to-\infty \tag{4.24}$$
and
$$\Lambda(\phi) \sim c_3 - c_1\phi - \phi^{\frac{1}{2}} - (2\pi\phi^{\frac{1}{2}})^{-1}[\ln\phi + \gamma + 1 + 2\ln 2] + o(\phi^{-\frac{1}{2}}) \quad \text{for} \quad \phi\to\infty. \tag{4.25}$$

Here c_1 is given by (4.21) while
$$c_2 = -(4\pi)^{-\frac{1}{2}}\int_0^\infty [(1+s^2)^{-\frac{1}{4}}e^{-H(s)} - 1]\,s^{-\frac{3}{2}}\,ds = 0 \tag{4.26}$$
and
$$c_3 = -(\tfrac{1}{2}\pi)^{\frac{1}{2}}\sin(\tfrac{3}{8}\pi) - (4\pi)^{-\frac{1}{2}}\int_0^\infty [(1+s^2)^{-\frac{1}{4}}e^{H(s)} - 1]\,s^{-\frac{3}{2}}\,ds = 0. \tag{4.27}$$

The zero values of c_2 and c_3 were obtained numerically, but we surmise that a proof, similar to that for c_1, which is based on (4.22), could be found to establish these results analytically.

5. Numerical results and discussion

Numerical results

Using (3.25) and (3.26), numerical values were obtained for $g(\phi)/|A_0|$ and
$$\Delta(\phi) \equiv f(\phi)/|A_0| - (\tfrac{1}{2}\pi)^{\frac{1}{2}}\sin(\phi+\tfrac{3}{8}\pi) \tag{5.1}$$

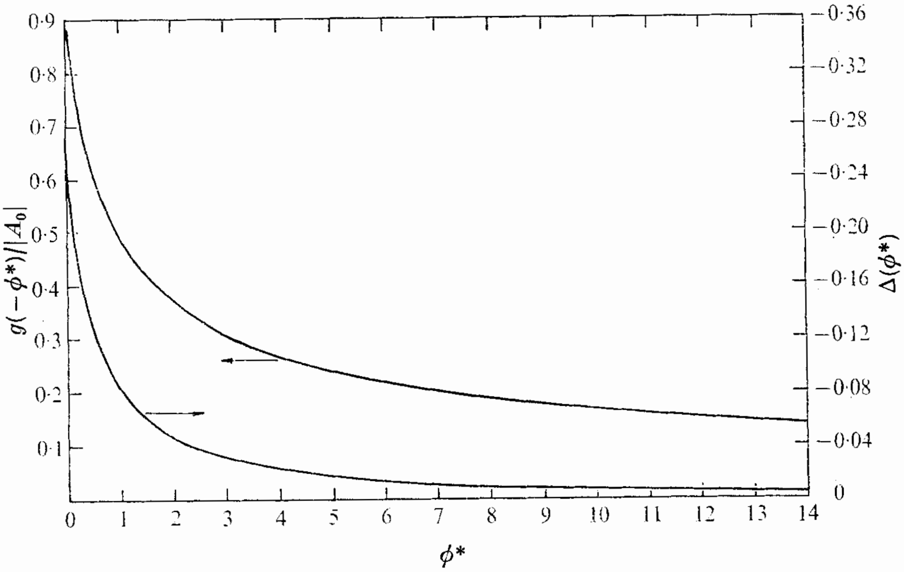

FIGURE 3. The functions $g(-\phi^*)/|A_0|$ and $\Delta(\phi^*)$ *vs.* ϕ^*. See (3.25), (3.26) and (5.1).

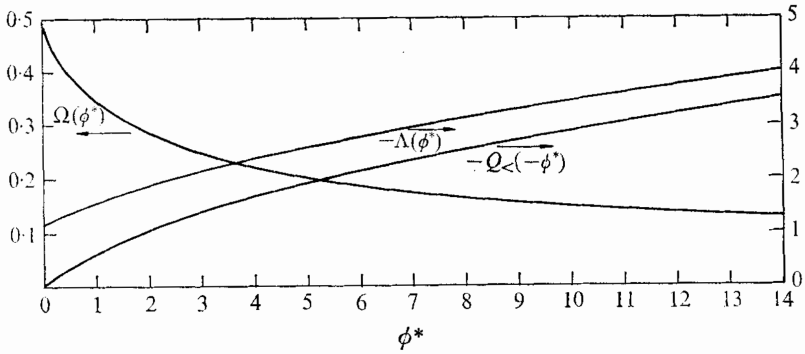

FIGURE 4. The functions $\Omega(\phi^*)$, $Q_<(-\phi^*)$ and $\Lambda(\phi^*)$ *vs.* ϕ^*. See (4.17), (4.18) and (4.20).

using the CDC 6600 computer at the Courant Institute. These results are displayed in figure 3. Results were also found for $\Omega(\phi)$, $Q_<(\phi)$ and $\Lambda(\phi)$, using (4.17), (4.18) and (4.20), and are shown in figure 4.

An equation for the free-streamline shape can be derived from (2.3). Introducing the change of variables (2.20) we find

$$dz/dw^* = \epsilon \exp\{-\epsilon^{\frac{1}{2}}\Gamma^*(w^*)\}. \tag{5.2}$$

Integrating along $\psi^* = 0$ and separating real and imaginary parts, we obtain

$$x/\epsilon = \int_0^{\phi^*} \exp\{-\epsilon^{\frac{1}{2}}Q_>(s)\}\cos[\epsilon^{\frac{1}{2}}\theta_>(s)]\,ds \tag{5.3}$$

and

$$y/\epsilon = \int_0^{\phi^*} \exp\{-\epsilon^{\frac{1}{2}}Q_>(s)\}\sin[\epsilon^{\frac{1}{2}}\theta_>(s)]\,ds.\dagger \tag{5.4}$$

† Note that y is $O(\epsilon^{\frac{3}{2}})$ for $\epsilon \to 0$ [see (2.24)].

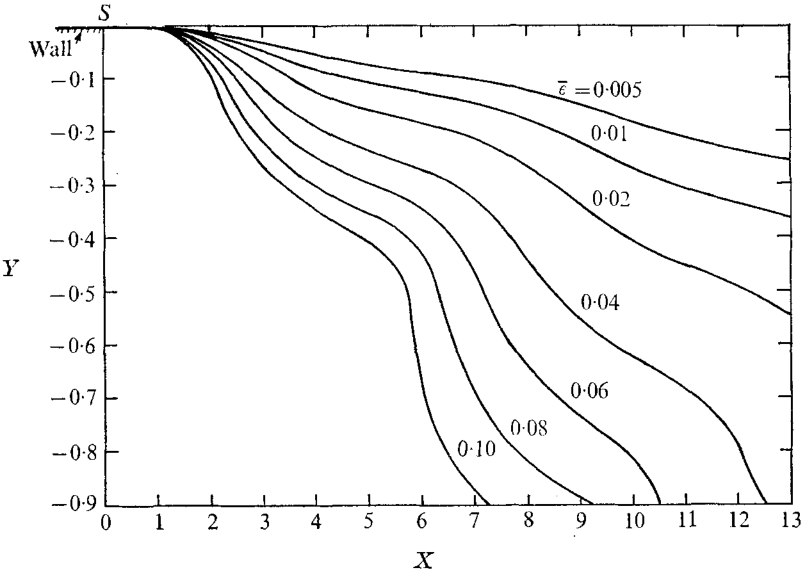

FIGURE 5. Free-streamline shapes in the inner region for various values of $\bar{\epsilon}$. See (5.3)–(5.5) and (5.7).

Since $Q_>$ and $\theta_>$ are linearly dependent on $|A_0|$, we may define the new parameter

$$\bar{\epsilon} = \epsilon|A_0|^2 \quad \text{with} \quad 0 < \bar{\epsilon} \ll 1, \tag{5.5}$$

and note that (5.3) and (5.4) define a functional relationship between x and y, with ϕ^* as parameter, which is of the form

$$Y = \mathscr{F}(X;\bar{\epsilon}), \tag{5.6}$$

where
$$X = |A_0|^2(x/\bar{\epsilon}), \quad Y = |A_0|^2(y/\bar{\epsilon}). \tag{5.7}$$

Numerical solutions of (5.3) and (5.4) were obtained for values of $\bar{\epsilon}$ in the range $0{\cdot}005 \leqslant \bar{\epsilon} \leqslant 0{\cdot}10$ and are displayed in figure 5. The capillary waves on the free surface are evident.

The coefficient of pressure C_p along the upstream wall can be defined by

$$C_p = (\bar{p}-p_0)/\tfrac{1}{2}\rho U_0^2 = 1-q^2-(\beta/2\alpha)y \quad \text{for} \quad \phi^* < 0. \tag{5.8}$$

If the wall is taken at $y = 0$, (5.8) may be written as

$$C_p(X) = 1-\exp\{2\bar{\epsilon}^{\frac{1}{2}}[Q_<(\phi^*)/|A_0|]\} \quad \text{on} \quad \psi^* = 0, \quad \phi^* < 0, \tag{5.9}$$

where
$$X = \int_0^{\phi^*} \exp\{-\bar{\epsilon}^{\frac{1}{2}}Q_<(s)/|A_0|\}\,ds \quad \text{on} \quad \psi^* = 0, \quad \phi^* < 0.$$

Numerical values for $C_p(X;\bar{\epsilon})$ are displayed in figure 6.

Table 1 lists the values of A_0 for two well-known free-streamline problems. These values were obtained from the solutions given in Lamb (1945, pp. 98 ff.).

Discussion of results

We summarize the important features of our solution, which are valid for large Weber numbers.

(i) The speed at the separation point is the same, to first order, with or without the effects of surface tension.

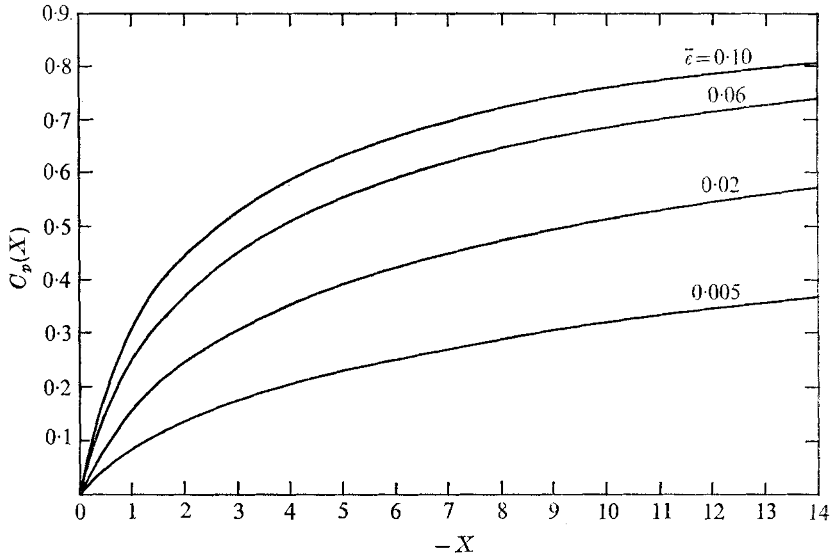

FIGURE 6. The coefficient of the pressure in the inner region along the upstream wall ($x < 0$, $y = 0$) for various values of $\bar{\epsilon}$. See (5.8) and (5.9).

Flow	L	U_0	A_0
Flat plate of breadth $2l$ set perpendicular to a uniform stream	$4l/(\pi+4)$	Separation-point speed	$-\sqrt{2}$
Flow from a two-dimensional slot of width d	$d/(\pi+2)$	Separation-point speed	$-\sqrt{2}$

TABLE 1. Values of A_0 with the appropriate characteristic lengths and velocities for two free-streamline problems

(ii) At the separation point the free streamline lies in the plane tangential to the wall at the separation point and has an *infinite* radius of curvature. Thus the radius of curvature of the $\psi = 0$ streamline will not be continuous at S unless the wall is perfectly flat. We note that the flow region in the physical plane will have dimensions $x = O(\bar{\epsilon}L)$ by $y = O(\bar{\epsilon}^{\frac{3}{2}}L)$, and within such a region a wall with a radius of curvature $O(L)$ in the outer variables appears to have a radius of curvature $O(L/\bar{\epsilon}^2) = \infty$, to first order when $\bar{\epsilon} \to 0$, in the inner variables. Therefore, in an asymptotic sense, the radius of curvature is continuous.

(iii) The free streamline has a wavelike shape as shown in figure 5. As $\bar{\epsilon}$ increases, the trend indicates a swelling which could make the jet, at certain points, extend upstream beyond the edge; this seems to indicate that some die swell phenomena may be partly accounted for by capillary wave action. For $\bar{\epsilon}$ not small, further work is necessary using the nonlinear boundary condition (2.9); this work is now under way.

(iv) The predicted singularities in the curvature and pressure gradient at the separation point which arise from the potential theory without surface tension are removed by a small non-zero surface-tension coefficient. As mentioned in (ii), the radius of curvature of the free streamline at S is infinite (for zero surface tension it is zero) and the pressure gradient at S in the physical (unscaled) variables (x, y) is favourable and $O(1/\bar{\epsilon}^{\frac{1}{2}})$ for $\bar{\epsilon} \rightarrow 0$. Observe that in figure 6 the abscissa is X, and using this variable in place of x, the pressure gradient will be favourable but $O(\bar{\epsilon}^{\frac{1}{2}})$. The boundary-layer separation studies of Ackerberg (1970, 1973*a*) will apply to flows with large Weber numbers if the boundary-layer solution is restricted to the region upstream of the edge along the wall where $x/L \gg O(R^{-\frac{4}{9}})$ provided that $R^{-\frac{4}{9}} \gg \bar{\epsilon}$, where R is the Reynolds number based on L. Further work on boundary-layer motions with capillary effects will be discussed in a subsequent paper.

(v) The concept of the 'static contact angle' appears to be irrelevant here, although it may be important when viscosity is taken into account. From our work this angle is always 180°, i.e. the liquid–gas interface is tangential to the wall as discussed in (ii); furthermore, this flow property is independent of the materials provided that the Weber number and Reynolds number are large. There may be other situations, of which we are not aware, where the non-uniqueness discussed in § 4 may be useful in constructing solutions which satisfy unusual boundary conditions at the edge. As an example, it might be desirable to suppress the standing wave on the free streamline by a proper combination of eigenfunctions.

(vi) The application of this work to axisymmetric potential flow separation with a non-zero surface-tension coefficient is of interest. Ackerberg (1973*b*) found that without surface-tension effects the axisymmetric flow separation in the azimuthal plane was the same as the two-dimensional flow to the first few orders. We now believe this to be true even with a small surface tension taken into account. The reason is that the boundary condition (2.8) along the free streamline would be modified by a transverse curvature term $O(L^{-1})$, but this is much smaller (i.e. of higher order) than the near-singular term which has already been included. Therefore, we should expect the streamline shapes in figure 5 to apply to the free streamlines in the azimuthal plane for axisymmetric flow. However, we do not know any values of A_0 for these cases owing to a lack of analytical solutions.

It would be interesting to know whether experimental observations near the edge would verify the existence of the predicted capillary waves. The theory is valid for large Weber numbers, and this will require observations in a very small region near the edge. We hope that the results presented here will stimulate experimentation.

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